On the Price of Truthfulness in Path Auctions

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Abstract. We study the frugality ratio of truthful mechanisms in path auctions, which measures the extent to which truthful mechanisms "overpay" compared to non-truthful mechanisms. In particular we consider the fundamental case that the graph is composed of two node-disjoint s-t-paths of length s_1 and s_2 respectively, and prove an optimal $\sqrt{s_1 s_2}$ lower bound (an improvement over $\sqrt{s_1 s_2/2}$). This implies that the $\sqrt{\ }$-mechanism of Karlin et al. for path auctions is 2-competitive (an improvement over $2\sqrt{2}$), and is optimal if the graph is a series-parallel network. Moreover, our results extend to universally truthful randomized mechanisms as well.

1 Introduction

Since the field of algorithmic mechanism design was introduced by Nisan and Ronen [NR99], *path auctions* have been studied extensively. In a path auction, the auctioneer tries to buy an s-t-path from a directed graph, where the edges of the graph are owned by *selfish* agents, and the cost of an edge is known *only* to its owner. *Truthful mechanisms*, the VCG mechanism [MCWG95] in particular, have been applied to path auctions. In such mechanisms, it is of each agent's best interest to simply report their private cost. However, as observed in [AT02, ESS04], every truthful mechanism can be forced to pay a high total amount to the agents. In contrast, the total payment is relatively small in first price non-truthful path auctions [IKNS05, CK07]. Such overpayment of truthful mechanisms compared to non-truthful mechanisms is seen as the *price of truthfulness* [KKT05], which we measure by the notion of *frugality ratio* of Karlin et al. [KKT05]. (The notion was actually proposed for all problems in the general hire-a-team setting [AT01], and Talwar also proposed a notion of frugality ratio with a different benchmark earlier in [Tal03].) Karlin et al. [KKT05] also proposed the $\sqrt{\ }$-mechanism for path auctions, which is $2\sqrt{2}$-competitive, i.e., by a factor of $2\sqrt{2}$ from optimal w.r.t. frugality ratio.

Behind many results on frugality ratio lies the fundamental case that the input graph $\mathcal{G}$ contains exactly two node-disjoint s-t-paths S_1, S_2 of length s_1 and s_2 respectively[1], which we call *1-out-of-2 (S_1, S_2)-auctions*. The VCG mechanism may overpay badly in this case, and has frugality ratio $\max\{s_1, s_2\}$. In

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[1] We use symbols S_i instead of P_i to be consistent with the notations in [KKT05].

[AT02], a $\frac{2s_1s_2}{s_1+s_2}$ lower bound was proved for the class of min-function (truthful) mechanisms, and later in [ESS04], a weaker $\frac{s_1s_2}{s_1+s_2}$ bound was obtained for all truthful mechanisms. Finally in [KKT05], a truthful mechanism with frugality ratio $\sqrt{s_1s_2}$ was proposed, and an asymptotic $\sqrt{s_1s_2/2}$ lower bound was proved as well, leaving a $\sqrt{2}$ gap open. But it is unlikely that this gap can be closed by previous proof methods, and our understanding of the overpayment issue in even this simple case is not complete.

Our Results In this paper, we introduce the interesting technique of *mechanism canonicalization*, and close the abovementioned gap by proving the following result, which to our knowledge is the first nontrivial tight lower bound known for frugality ratios. Moreover, this result can be extended to universally truthful randomized mechanisms [NR99] as well.

Theorem 1. *The frugality ratio of 1-out-of-2 (S_1, S_2)-auctions is $\Phi_{s_1,s_2} = \sqrt{s_1s_2}$.*

1-out-of-2 auctions are embedded in not only path auctions, but also many other problems, including vertex cover [EGG07], minimum cost bipartite matching etc. It follows that lower bounds about 1-out-of-2 auctions extend to those problems by reductions. In particular, for path auctions:

Theorem 2. *The $\sqrt{\ }$-mechanism for path auctions is 2-competitive in general, and is optimal if the input graph is a series-parallel network.*

2 The Model

In the setting, $\mathcal{G} = (V, E)$ is a directed graph where V contains two fixed vertices s and t. Each edge e in E represents a *selfish* agent, and has a privately known nonnegative cost c_e[2], which occurs if the agent is selected. A *path auction* consists of two steps. First each agent e submits a sealed bid b_e to the auctioneer. Then based on the bids, the auctioneer applies a *selection rule* to select an s-t-path P as the winning path, and pays an amount $p_e \geq b_e$ to each agent in P. We say that the agents in P win, and the others lose. The selection rule and payment rule together constitute a *mechanism* for $\mathcal{G}$. We assume that each agent is *rational*, fully knows about $\mathcal{G}$ and the mechanism, and aims at maximizing his own profit, which is $p_e - c_e$ if he wins, and 0 otherwise. As is standard, we assume that $\mathcal{G}$ has no s-t cut edge, otherwise there would be a monopoly.

We say that a mechanism is *truthful*, if each agent e can maximize his profit by bidding his true cost c_e, i.e., $b_e = c_e$, no matter what the others bid. There are two characteristic properties about truthful mechanisms: [AT01, AT02]

The Monotonicity Property If a mechanism is *truthful*, then the associated selection rule is *monotone*, i.e., a winning agent still wins if he decreases his bid, given fixed bids of the others.

[2] For costs, bids, etc., we extend the notation by writing $c(T)$ for $\sum_{e \in T} c_e$, etc.

The Threshold Property Given a monotone selection rule, there is a *unique* truthful mechanism associated with this selection rule. Moreover, this mechanism pays each agent the *threshold bid*, i.e., the supremum of the amounts that the agent can still win by bidding, given fixed bids of the others.

Let $\mathcal{M}$ be a truthful mechanism for $\mathcal{G}$. Given the cost vector $\mathbf{c}$ (or equivalently, bid vector, since $\mathcal{M}$ is truthful) of the agents, let $p_{\mathcal{M}}(\mathbf{c})$ denote the total payment made by $\mathcal{M}$ to the agents. We use $\nu(\mathbf{c})$ to denote the benchmark for overpayment, whose definition we omit here. But in 1-out-of-2 (S_1, S_2)-auctions, $\nu(\mathbf{c})$ simply equals to the maximum of $c(S_1)$ and $c(S_2)$. The *frugality ratio* $\phi_{\mathcal{M}}$ of a truthful mechanism $\mathcal{M}$ is $\sup_{\mathbf{c}\neq\mathbf{0}} \rho_{\mathcal{M}}(\mathbf{c})$, where $\rho_{\mathcal{M}}(\mathbf{c}) = p_{\mathcal{M}}(\mathbf{c})/\nu(\mathbf{c})$, and the frugality ratio $\Phi_{\mathcal{G}}$ of a graph $\mathcal{G}$, or the path auction on $\mathcal{G}$, is the infimum of $\phi_{\mathcal{M}}$ over all truthful mechanisms for $\mathcal{G}$.

3 1-out-of-2 Auctions

In this section, we show that the frugality ratio Φ_{s_1,s_2} of 1-out-of-2 (S_1, S_2)-auctions is exactly $\sqrt{s_1 s_2}$. For brevity, every mechanism we mention here is a truthful mechanism for 1-out-of-2 (S_1, S_2)-auctions. Consider the mechanism $\mathcal{M}$ such that the S_i with the least value of $\sqrt{s_i} \cdot c(S_i)$ is selected from $i = 1, 2$ with ties broken arbitrarily. One can verify that $\Phi_{\mathcal{M}} \leq \sqrt{s_1 s_2}$. To see this, let the costs of the agents be $\mathbf{c}$, and w.l.o.g. let S_1 wins. Then the threshold bid of each agent $e \in S_1$ is at most $\sqrt{s_2} \cdot c(S_2)/\sqrt{s_1}$. So $\rho_{\mathcal{M}}(\mathbf{c}) \leq p_{\mathcal{M}}(\mathbf{c})/\nu(\mathbf{c}) \leq \sqrt{s_1 s_2} \cdot c(S_2)/\nu(\mathbf{c}) \leq \sqrt{s_1 s_2}$, and hence $\Phi_{s_1,s_2} \leq \sqrt{s_1 s_2}$. We devote the rest of this section to lower bound.

To fix some conventions, we use R_+ to denote the set of nonnegative reals. If $\mathbf{w}$ is a vector in R_+^n, then w_i denotes its ith component. A vector function $\mathbf{t}$: $R_+^m \to R_+^n$ is seen as an n-tuple of functions t_j: $R_+^m \to R_+$ for $1 \leq j \leq n$. We say that vector $\mathbf{w} \in R_+^n$ is *dominated* by vector $\mathbf{w}' \in R_+^n$, or write $\mathbf{w} \preceq \mathbf{w}'$, if $w_i \leq w_i'$ for all i. We let $\mathbf{e}_i$ denote the unit vector with the ith component 1 and others 0. Agents in S_i are numbered from 1 to s_i for $i = 1, 2$. We say that S_i wins at $(\mathbf{u}, \mathbf{v})$ if S_i is selected when the costs $\mathbf{c}$ of the agents are $(\mathbf{u}, \mathbf{v})$, where each u_i is the cost of agent i in S_1, and each v_j is the cost of agent j in S_2. We also assume w.l.o.g. that S_2 wins at $(\mathbf{u}, \mathbf{0})$ if $\mathbf{u} \neq \mathbf{0}$ and S_1 wins at $(\mathbf{0}, \mathbf{v})$ if $\mathbf{v} \neq \mathbf{0}$.

3.1 The $\mathbf{t}^{\mathcal{M}}$ Function

For a mechanism $\mathcal{M}$, function $\mathbf{t}^{\mathcal{M}}$: $R_+^{s_1} \to R_+^{s_2}$ is defined as $t_j^{\mathcal{M}}(\mathbf{u}) = \sup\{y\colon S_2$ wins at $(\mathbf{u}, y\mathbf{e}_j)\}$, for all $\mathbf{u} \in R_+^{s_1}$ and $1 \leq j \leq s_2$.[3] We find the following way of visualization helpful. Let $s_2 = 2$, and refer to Fig. 1. The solid curve indicates the boundary between the area where S_2 wins and the area where S_1 wins. (If $s_2 > 2$, then the boundary is a surface instead.) By the monotonicity property,

[3] We may drop the superscript $\mathcal{M}$ when the context is clear.

loosely speaking, the boundary monotonically decreases. Also by the threshold property, if S_2 wins at $(\mathbf{u}, \mathbf{v})$, the payment to the agents in S_2 is the total length of the two segments crossing at $\mathbf{v}$. With such intuition, it is easy to observe the following properties.

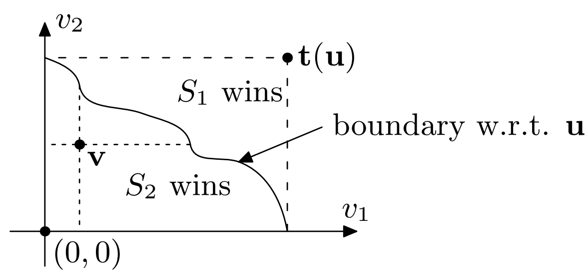

Fig. 1.

Lemma 1. (i) *If S_2 wins at $(\mathbf{u}, \mathbf{v})$, then $\mathbf{v} \preceq \mathbf{t}(\mathbf{u})$. In addition, agent j in S_2 is paid at most $t_j(\mathbf{u})$ for all $1 \le j \le s_2$.*

(ii) *If $\mathbf{u} \preceq \mathbf{u}'$ then $\mathbf{t}(\mathbf{u}) \preceq \mathbf{t}(\mathbf{u}')$. I.e., $\mathbf{t}$ respects the dominance relation.*

(iii) *For a mechanism $\mathcal{M}$, $\phi_{\mathcal{M}}$ equals to the maximum of $\sup_{\mathbf{u}\neq\mathbf{0}} \rho_{\mathcal{M}}(\mathbf{u}, \mathbf{0})$ and $\sup_{\mathbf{v}\neq\mathbf{0}} \rho_{\mathcal{M}}(\mathbf{0}, \mathbf{v})$. In addition, $\rho_{\mathcal{M}}(\mathbf{u}, \mathbf{0})$ equals to $\sum_{j=1}^{s_2} t_j(\mathbf{u}) / \sum_{i=1}^{s_1} u_i$ and $\rho_{\mathcal{M}}(\mathbf{0}, \mathbf{v})$ equals to $\sum_{i=1}^{s_1} \sup\{x\text{: } S_1 \text{ wins at } (x\mathbf{e}_i, \mathbf{v})\} / \sum_{j=1}^{s_2} v_j$.*

3.2 Mechanism Canonicalization

For each mechanism $\mathcal{M}$, in the following we canonicalize $\mathcal{M}$ into a type-1 mechanism $\mathcal{M}_1$, and then into a type-2 mechanism $\mathcal{M}_2$, and finally into a type-3 mechanism $\mathcal{M}_3$ respectively. In the process, frugality is preserved, i.e., $\phi_{\mathcal{M}} \ge \phi_{\mathcal{M}_1} \ge \phi_{\mathcal{M}_2} \ge \phi_{\mathcal{M}_3}$. It follows that Φ_{s_1,s_2} can be determined by analyzing the infimum of $\phi_{\mathcal{M}_3}$ over all type-3 mechanism $\mathcal{M}_3$, while the special properties of the class of type-3 mechanisms can be taken advantage of in the analysis. We call such technique as mechanism canonicalization.

Type-1 Mechanisms For a mechanism $\mathcal{M}$, we first canonicalize it into the mechanism $\mathcal{M}_1$ such that S_2 wins at $(\mathbf{u}, \mathbf{v})$ in $\mathcal{M}_1$ iff $\mathbf{v} \preceq \mathbf{t}^{\mathcal{M}}(\mathbf{u})$. One can verify that the selection rule of $\mathcal{M}_1$ is monotone, and such canonicalized mechanisms are called *type-1 mechanisms*. In particular, it is guaranteed that $\phi_{\mathcal{M}_1} \le \phi_{\mathcal{M}}$. To verify this via Lemma 1(iii), we need to show that $\sup\{x\text{: } S_1$ wins at $(x\mathbf{e}_i, \mathbf{v})$ in $\mathcal{M}_1\} \le \sup\{x : S_1$ wins at $(x\mathbf{e}_i, \mathbf{v})$ in $\mathcal{M}\}$. This is true because if S_2 wins at $(x\mathbf{e}_i, \mathbf{v})$ in $\mathcal{M}$ for some $i, x, \mathbf{v}$, then $\mathbf{v} \preceq \mathbf{t}^{\mathcal{M}}(x\mathbf{e}_i)$ by Lemma 1(i), and then by the definition of $\mathcal{M}_1$, S_2 wins at $(x\mathbf{e}_i, \mathbf{v})$ in $\mathcal{M}_1$ too.

The following follows directly from Lemma 1(iii).

Lemma 2. *Let $\mathcal{M}_1$ be a type-1 mechanism. Then $\phi_{\mathcal{M}_1} \leq r$ if and only if conditions (a) and (b) hold:*

(a) For all $\mathbf{u} \neq \mathbf{0}$, $\rho_{\mathcal{M}_1}(\mathbf{u}, \mathbf{0}) = \sum_j t_j(\mathbf{u}) / \sum_i u_i \leq r$.
(b) For all $\mathbf{v} \neq \mathbf{0}$, $\rho_{\mathcal{M}_1}(\mathbf{0}, \mathbf{v}) = \sum_i \sup\{x : \mathbf{v} \not\preceq \mathbf{t}(x\mathbf{e}_i)\} / \sum_j v_j \leq r$.

Type-2 Mechanisms Note that each type-1 mechanism $\mathcal{M}_1$ can be determined by its $\mathbf{t}^{\mathcal{M}_1}$ function (denoted by $\mathbf{t}$ for brevity) . Consider the $\hat{\mathbf{t}}$ function such that for all $\mathbf{u}$ in the form of $u_i\mathbf{e}_i$ for some i, $\hat{\mathbf{t}}(u_i\mathbf{e}_i) = \mathbf{t}(u_i\mathbf{e}_i)$, and for all other $\mathbf{u}$, $\hat{\mathbf{t}}(\mathbf{u}) = \sum_i \hat{\mathbf{t}}(u_i\mathbf{e}_i)$. We then canonicalize $\mathcal{M}_1$ into $\mathcal{M}_2$, which is the type-1 mechanism determined by the $\hat{\mathbf{t}}$ function, i.e., $\mathbf{t}^{\mathcal{M}_2} = \hat{\mathbf{t}}$. Each such $\mathcal{M}_2$ is called a *type-2 mechanism*. One can verify that $\hat{\mathbf{t}}$ respects the dominance relation, and therefore the selection rule of $\mathcal{M}_2$ is monotone. Clearly type-2 mechanisms are determined by their $\mathbf{t}(u_i\mathbf{e}_i)$ functions, and it turns out that Lemma 2 can be correspondingly refined to the following.

Lemma 3. *Let $\mathcal{M}_2$ be a type-2 mechanism. Then $\phi_{\mathcal{M}_2} \leq r$ if and only if the following conditions hold:*

(a') *For all $u_i \neq 0$ and i, $\rho_{\mathcal{M}_2}(u_i\mathbf{e}_i, \mathbf{0}) = \sum_j t_j(u_i\mathbf{e}_i)/u_i \leq r$.*
(b') *For all $v_j \neq 0$ and j, $\rho_{\mathcal{M}_2}(\mathbf{0}, v_j\mathbf{e}_j) = \sum_i \sup\{x : t_j(x\mathbf{e}_i) < v_j\}/v_j \leq r$.*

Note that $\mathbf{t}^{\mathcal{M}_2}(u_i\mathbf{e}_i) = \mathbf{t}^{\mathcal{M}_1}(u_i\mathbf{e}_i)$ for all u_i and i, and so by Lemma 3, we have $\phi_{\mathcal{M}_2} = \phi_{\mathcal{M}_1}$.

Type-3 Mechanisms In a type-2 mechanism $\mathcal{M}_2$, if each $\mathbf{t}^{\mathcal{M}_2}(x\mathbf{e}_i)$ function (denoted by $\mathbf{t}(x\mathbf{e}_i)$ for brevity) is a *curve*, i.e., a continuous mapping from R_+ to $R_+^{s_2}$, we say that it is a *type-3 mechanism.*

Lemma 4. *For each type-2 mechanism $\mathcal{M}_2$, there is a canonicalized type-3 mechanism $\mathcal{M}_3$ such that $\phi_{\mathcal{M}_3} = \phi_{\mathcal{M}_2}$.*

3.3 Determining $\boldsymbol{\Phi}_{s_1,s_2}$

Based on the canonicalization process, $\Phi_{s_1,s_2} \leq r$ is equivalent to that there is a type-3 mechanism $\mathcal{M}_3$ with $\phi_{\mathcal{M}_3} \leq r$. Since a type-3 mechanism is determined by its $\mathbf{t}(x\mathbf{e}_i)$ functions, this equivalence can be rephrased as follows: (with each $\mathbf{t}(x\mathbf{e}_i)$ renamed to $\mathbf{g}^i$)

Theorem 3. *$\Phi_{s_1,s_2} \leq r$ if and only if there exist curves $\mathbf{g}^1, \ldots, \mathbf{g}^{s_1}\colon R_+ \to R_+^{s_2}$ such that the following conditions are satisfied:*

(i) $\mathbf{g}^i(x) \preceq \mathbf{g}^i(x')$ for all $1 \leq i \leq s_1$ and $x \leq x'$.
(ii) $\sum_j g_j^i(x) \leq xr$ for all $1 \leq i \leq s_1$ and x.
(iii) $\sum_i \sup\{x : g_j^i(x) \leq y\} \leq yr$ for all $1 \leq j \leq s_2$ and y.

So the problem of determining Φ_{s_1,s_2} is converted to an equivalent pure math problem about curves, which can be solved by applying the Young's Inequality.

Proof. (of Theorem 1) First note that one can prove that $\Phi_{s_1,s_2} \leq \sqrt{s_1 s_2}$ via Theorem 3 by setting $r = \sqrt{s_1 s_2}$ and $g^i_j(x) = \sqrt{s_1/s_2} \cdot x$ for all i, j. To prove that $\Phi_{s_1,s_2} \geq \sqrt{s_1 s_2}$, let $\mathbf{g}^1, \ldots, \mathbf{g}^{s_1}$ and r satisfy the conditions in Theorem 3. By (b) of Theorem 3, $\sum_{j'} g^i_{j'}(g^i_{-j}(y))/r \leq g^i_{-j}(y)$ for all i, j. Add a summation over i, and then by (c), $\sum_i \sum_{j'} g^i_{j'}(g^i_{-j}(y))/r \leq \sum_i g^i_{-j}(y) \leq yr$, for all j. Denote $g^i_{j'}(g^i_{-j}(y))$ by $h^i_{j\to j'}(y)$, add another summation over j, and we have $\sum_i \sum_j \sum_{j'} h^i_{j\to j'}(y) \leq s_2 r^2 y$. Note that each $h^i_{j\to j'}(y)$ is increasing, and hence we can define its integral function: $H^i_{j\to j'}(y) = \int_0^y h^i_{j\to j'}(z)dz$, for all i, j, j'. Assume for simplicity that each $h^i_{j\to j'}$ is monotone. Then by applying the Young's Inequality, $H^i_{j\to j'}(y)+H^i_{j'\to j}(y) \geq y^2$ for all i, j, j'. So $\frac{1}{2}s_2 r^2 y^2 \geq \sum_i \sum_j \sum_{j'} H^i_{j\to j'}(y) \geq \frac{1}{2}s_1 s_2^2 y^2$, and thus $r \geq \sqrt{s_1 s_2}$. It follows that $\Phi_{s_1,s_2} \geq \sqrt{s_1 s_2}$ by Theorem 3.

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